Resource Letter VWCPF-1: Van der Waals and Casimir-Polder forces

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Abstract

This Resource Letter provides a guide to the literature on van der Waals and Casimir-Polder forces. Journal articles, books, and other documents are cited on the following topics: nonretarded or van der Waals forces, retarded dispersion forces or Casimir-Polder forces between atoms or molecules, Casimir-Polder forces between a molecule and a dielectric or conducting body, the summation of Casimir-Polder forces as leading to the Casimir and Lifshitz forces between conducting and dielectric bodies, Casimir friction, applications to nanotechnology, the nature of the quantum vacuum, and experimental tests of the theory of Casimir and Casimir-Polder and van der Waals forces.

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I. INTRODUCTION

The forces between molecules, first inferred from deviations from the ideal gas laws, are often due to quantum fluctuations. These can be thought of as fluctuations of the molecular or atomic electric-dipole moments themselves, either permanent or induced, or as fluctuations in the electromagnetic fields surrounding the dipoles. These forces were calculated from first principles starting in the 1930s in the nonretarded regime (that is, the finiteness of the speed of light being neglected) where the forces are called van der Waals, and in 1947 in the retarded regime (for distances large compared to the speed of light times a characteristic time) where they are referred to as Casimir-Polder (CP) forces. Progress in the subject has continued at an accelerating pace since then. The applications are important in atomic and molecular physics, quantum field theory, and nanotechnology. For example, Casimir attraction can give rise to "stiction" (frictional forces that cause parts to stick together) between the micro-components, which will cause the device to fail, or, more excitingly, Casimir forces can drive the operation of nanomachinery. Van der Waals and Casimir-Polder forces may be thought of as special cases of, or as the origin of the general Casimir or quantum vacuum forces between macroscopic and mesoscopic bodies, and provide an important window into the quantum fluctuating nature of reality.

It should be noted that the author has been highly selective in choosing representative articles in this rapidly developing field. Because the central focus is on van der Waals and Casimir-Polder forces involving atoms, many topics of great importance in the general field of quantum vacuum energy phenomena, such as the dynamical Casimir effect, the coupling of Casimir energy to gravity or to modified gravity, and generally, Casimir energies in curved spacetime, have been excluded.

In this Resource Letter we use Gaussian units. To convert to SI units, replace the Gaussian polarizability α by $\alpha^{\text{SI}}/4\pi\epsilon_0$. and replace the Gaussian permittivity ε by the relative permittivity $\varepsilon^{\text{SI}}/\epsilon_0$.

II. BOOKS, POPULAR ARTICLES, AND REVIEWS

Quantum vacuum forces are discussed in the following books and monographs.

1. **The Quantum Vacuum: An Introduction to Quantum Electrodynamics**, P. W. Milonni (Academic Press, Boston, 1994). (I)

2. **The Casimir Effect and Its Applications**, V. M. Mostepanenko and N. N. Trunov (Oxford University Press, Oxford, 1997). (A)

3. **The Casimir Effect: Physical Manifestations of Zero-Point Energy**, K. A. Milton (World Scientific, Singapore, 2001). (A)

4. **Advances in the Casimir Effect**, M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko (Oxford University Press, Oxford, 2009). (A)

Somewhat more specialized are books with a molecular orientation:

5. **Dispersion Forces**, J. Mahanty and B. W. Ninham (Academic Press, London, 1976). (I)

6. **Molecular Quantum Electrodynamics: An Introduction to Radiation-Molecule Interactions**, D. P. Craig and T. Thirunamachandran (Academic Press, London. 1984). (I)

7. **Intermolecular and Surface Forces**, J. Israelachvili (Academic Press, New York, 1985). (I)

8. **Molecular Quantum Electrodynamics: Long-range Intermolecular Interactions**, A. Salam (Wiley, Hoboken, NJ, 2010). (I)

Very useful books of a more mathematical character are the following:

9. **Zeta Regulation Techniques with Applications**, E. Elizalde, S. D. Odintsov, and A. Romeo (World Scientific, Singapore, 1994). (A)

10. **Ten Physical Applications of Spectral Zeta Functions** (Lecture Notes in Physics), E. Elizalde (Springer, Berlin, 1995). (A)

11. **Spectral Functions in Mathematics and Physics**, K. Kirsten (Chapman & Hall/CRC Press, Boca Raton, FL, 2002).

The following are a selection of review articles:

12. "The Casimir Effect," G. Plunien, B. Müller, and W. Greiner, Phys. Rep. **134**, 87–193 (1986). (I)

13. "The Casimir Effect and Its Applications," V. M. Mostepanenko and N. N. Trunov, Sov. Phys. Usp. **31**, 965–987 (1988). (I)

14. "New Developments in the Casimir Effect," M. Bordag, U. Mohideen, and V. M. Mostepanenko, Phys. Rept. **353**, 1–205 (2001). (A)

15. "The Casimir Effect: Recent Controversies and Progress," K. A. Milton, J. Phys. A **37**, R209–R277 (2004). (A)

16. "Dispersion Forces in Macroscopic Quantum Electrodynamics." S. Y. Buhmann and D.-G. Welsch, Prog. Quant. Electron. **31**, 51–130 (2007). (I)

17. "The Casimir Force Between Real Materials: Experiment and Theory," G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, Rev. Mod. Phys. **81**, 1827–1885 (2009). (A)

A Resource Letter on the Casimir Effect was published over a decade ago.

18. "Resource Letter CF-1: Casimir Force," S. K. Lamoreaux, Am. J. Phys. **67**, 850–861 (1999). (E)

The following give historical perspective:

19. **Physics in the Making: Essays on Developments in 20th Century Physics in Honour of H. B. G. Casimir on the Occasion of his 80th Birthday**, A. Sarlemijn and M. J. Spaarnay (North Holland, Amsterdam, 1989). (E)

20. "Some Remarks on the History of the So-Called Casimir Effect," H. B. G. Casimir, in **Casimir Effect 50 Years Later**, ed. M. Bordag (World Scientific, Singapore 1999), pp. 3–9 [Proceedings of the Fourth Workshop on Quantum Field Theory Under the Influence of External Conditions, Leipzig, 1998]. (E)

21. "Casimir Effects in Atomic, Molecular, and Optical Physics," J. F. Babb, Advances in Atomic, Molecular, and Optical Physics **59**, 1–20 (2010). (E)

The QFEXT series of conferences give a continuing overview. The latest proceedings is the following.

22. **Proceedings of the Ninth Conference on Quantum Field Theory Under the Influence of External Conditions (QFEXT09)**, ed. K. A. Milton and M. Bordag (World Scientific, Singapore, 2010). (A)

Other recent collections of papers include the following:

23. **Cosmology, the Quantum Vacuum, and Zeta Functions: A Choice of Papers**, eds. E. Elizalde and S. D. Odintsov (Tomsk State Pedagogical University, Tomsk, Russia, 2009). (A)

24. **Cosmology, Quantum Vacuum, and Zeta Functions**, eds. D. Sáez-Gómez, S. D. Odintsov, and S. Xambó (Springer, Berlin, 2011). (A)

Popular articles include:

25. "The Force Between Molecules," B. V. Derjaguin, Sci. Am. **203** (7), 47–53 (1960). (E)

26. "Retarded, or Casimir, Long-Range Potentials," L. Spruch, Physics Today **39** (11) 37–45 (1986). (E)

27. "Essentials of the Casimir Effect and its Computation," E. Elizalde and A. Romeo, Am. J. Phys. **59**, 711–719 (1991). (I)

28. "Long Range (Casimir) Interactions," L. Spruch, Science **272**, 1452–1455 (1996). (E)

29. "A Maritime Analogy of the Casimir Effect," S. L. Boersma, Am. J. Phys. **64**, 539–541 (1996). (E)

30. "Experiment and Theory in the Casimir Effect," G. L. Klimchitskaya and V. M. Mostepanenko, Contemp. Phys. **47**, 131–144 (2006). (E)

31. "Casimir Forces: Still Surprising After 60 Years," S. K. Lamoreaux, Physics Today **60** (2), 40–45 (2007). (E)

32. "Casimir Effects," P. W. Milonni, Phys. Scr. **76**, C167–C171 (2007). (E)

33. "Casimir Force Could Drive Tiny Ratchets," H. Johnston, Physics World, May 2, 2007. (E)

34. "Casimir Effect Goes Negative," E. Cartlidge, Physics World, Jan. 8, 2009. (E)

35. "Casimir Forces between Solids Can Be Repulsive," J. Miller, Physics Today **62** (2), 19–21. (E)

36. "Thermal Casimir Force Seen for the First Time," H. Johnston, Physics World, Feb. 8, 2011. (E)

III. VAN DER WAALS TO CASIMIR-POLDER

Van der Waals forces between molecules were first inferred from deviations from the ideal gas laws.

37. **O ver de Continueïteit van den Gas- en Vloeistoftocstand**, J. D. van der Waals (Ph.D. thesis, Leiden, 1873). (E)

38. "O ver de Continuiteit van den Gas- en Vloeistofiocstand Academisch Proefschrift" (book review), J. C. Maxwell, Nature **10**, 477–480 (1874). (I)

But quantum mechanics was required to understand how such forces could arise from neutral atoms or molecules. In the short-distance, or nonretarded regime, where the speed of light is regarded as infinite, London derived, using second-order perturbation theory, the interaction between two fluctuating dipoles, characterized by a polarizability α. That is, the dipole moment $\mathbf{p}$ is proportional to the electric field $\mathbf{E}$, $\mathbf{p}(\omega) = \alpha(\omega)\mathbf{E}(\omega)$. The resulting interaction energy fell off with the distance r between the atoms like r^{-6},

$$V_{\text{vdW}} = -\frac{3}{\pi}\frac{\hbar}{r^6}\int_0^\infty d\zeta\,\alpha_1(i\zeta)\alpha_2(i\zeta), \tag{3.1}$$

where α_i is the polarizability of the ith atom, as a function of the imaginary frequency $\omega = i\zeta$ (that is, we rotate ω to the imaginary axis, so ζ is real).

39. "Zur Theorie und Systematik der Molekularkräfte," F. London, Z. Physik **63**, 245–279 (1930). (I)

40. "Über das Verhältnis der Waalsschen Kräfte zu den homöopolarn Bindungskräften," R. Eisenschitz and F. London, Z. Phys. **60**, 491–527 (1930). (I)

This, however, only holds for distances of order 10 nm or less. For longer distances, the finite speed of light must be taken into account, that is, retardation must be considered. Casimir and Polder showed that the interaction energy between two isotropic polarizable atoms falls off faster, like r^{-7}, at zero temperature, in terms of the static, zero-frequency, polarizabilities $\alpha_i = \alpha_i(0)$,

$$V_{\mathrm{CP}} = -\frac{23}{4\pi}\frac{\alpha_1\alpha_2}{r^6}\frac{\hbar c}{r}, \tag{3.2a}$$

whereas, as was later realized, for high temperature, the dependence on $\hbar c/r$ is replaced by kT, where k is Boltzmann's constant,

$$V_{\mathrm{CP}} \sim -\frac{\alpha_1\alpha_2}{r^6}kT, \quad T \to \infty. \tag{3.2b}$$

Casimir and Polder reproduced the London result at short distances, where the transition between the retarded and nonretarded regimes is set by some characteristic wavelengths of the relevant atomic transitions.

41. "The Influence of Retardation on the London-van der Waals Forces," H. B. G. Casimir and D. Polder, Phys. Rev. **73**, 360–372 (1948). (I)

In this paper, the authors also derived the interaction energy between an atom, of static polarizability α, and a metallic plate a distance Z away,

$$E_{\mathrm{CP}} = -\frac{3\alpha}{8\pi Z^4}\hbar c. \tag{3.3}$$

As a result of a conversation with Bohr, Casimir realized that these results could be obtained more simply by considering the zero-point fluctuations of the electromagnetic fields between the atoms, or the atom and the plate, as he explained in a lecture in Paris:

42. "Sur les Forces van der Waals-London," H. B. G. Casimir, in **Colloque sur la théorie de la liaison chimique**, Paris, 12–17 April 1948, published in J. Chim. Phys. et Phys. Chim. Biol. **46**, 407–410 (1949). (I)

If the atoms are not isotropic (spherically symmetric) the interaction depends on the orientation between the polarization tensors of the atoms. Thus, if we assume only linearity, so that there is a tensor (dyadic) polarization,

$$\mathbf{p} = \boldsymbol{\alpha} \cdot \mathbf{E}, \tag{3.4}$$

the interaction between such an atom and an isotropic slab a distance Z away is generalized to

$$E_{\text{CP}} = -\frac{\text{Tr } \boldsymbol{\alpha}}{8\pi Z^4}. \tag{3.5}$$

And the Casimir-Polder potential between two anisotropic atoms is generalized to

$$V_{\text{CP}} = -\frac{1}{4\pi r^7} \left[\frac{13}{2} \text{Tr } \boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2 - 28 \text{Tr} \left(\boldsymbol{\alpha}_1 \cdot \hat{\mathbf{r}} \right) \left(\boldsymbol{\alpha}_2 \cdot \hat{\mathbf{r}} \right) + \frac{63}{2} (\hat{\mathbf{r}} \cdot \boldsymbol{\alpha}_1 \cdot \hat{\mathbf{r}}) (\hat{\mathbf{r}} \cdot \boldsymbol{\alpha}_2 \cdot \hat{\mathbf{r}}) \right], \tag{3.6}$$

where $\mathbf{r} = r\hat{\mathbf{r}}$ is the relative position vector of the two atoms, and Tr stands for the trace of the matrix. This generalizes the form given in Ref. [41] when the two polarizations are not simultaneously diagonalizable.

For a pedagogical non-field-theoretical rederivation of van der Waals and Casimir-Polder forces, see

43. "The van der Waals Interaction," B. R. Holstein, Am. J. Phys. **69**, 441–449 (2001). (I)

See also Sec. VI. For interactions between a polarizable atom and a multilayer substrate, see

44. "van der Waals and Retardation (Casimir) Interactions of an Electron or an Atom with Multilayered Walls," F. Zhou and L. Spruch, Phys. Rev. A **52**, 297–310 (1995). (I)

IV.   CASIMIR EFFECT

The realization by Casimir that Casimir-Polder forces could be understood in terms of quantum field fluctuations allowed him to immediately recognize that there would be a quantum-electrodynamic interaction between two neutral parallel conducting plates, which resulted in an attractive force F between those plates having area A and separation distance a:

$$\frac{F}{A} = -\frac{\pi^2 \hbar c}{240 a^4}. \tag{4.1}$$

45. "On the Attraction Between Two Perfectly Conducting Plates," H. B. G. Casimir, Proc. Kon. Ned. Akad. Wetensch. **51**, 793–795 (1948). (I)

Numerically, this force is quite small,

$$\frac{F}{A} = -1.30 \times 10^{-27} \text{ N } \text{m}^2 a^{-4}, \tag{4.2}$$

but quite significant at the 100 nm scale.

A few years later, Lifshitz generalized the Casimir force between perfectly conducting plates to the force between dielectric slabs.

46. "The Theory of Molecular Attractive Forces Between Solids," E. M. Lifshitz, Zh. Eksp. Teor. Fiz., **29**, 94–110 (1955) [English transl.: Soviet Phys. JETP **2**, 73–83 (1956)]. (I)

47. "General Theory of van der Waals' Forces," I. D. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, Usp. Fiz. Nauk **73**, 381–422 (1961) [English transl.: Soviet Phys. Usp. **4**, 153–176 (1961)]. (A)

48. **Electrodynamics of Continuous Media**, L. D. Landau and E. M. Lifshitz (Pergamon, Oxford, 1960). (I)

The Lifshitz formula may be easily understood in terms of multiple reflections between the parallel surfaces; an accessible derivation is found in Ref. [15]. For the case of two thick parallel slabs, with dielectric constants ε_1, ε_2, respectively, separated by a medium having dielectric constant ε_3 and thickness a, the force/area at zero temperature is given in terms of integrals over imaginary frequency ζ and wave number k (both with dimensions of inverse length):

$$\frac{F}{A} = -\frac{\hbar c}{8\pi^2} \int_0^\infty d\zeta \int_0^\infty dk^2 2\kappa_3 \left(d_{\text{TE}}^{-1} + d_{\text{TM}}^{-1} \right), \tag{4.3}$$

where for the "transverse electric" polarization

$$d_{\text{TE}} = \left(r_{13}^{\text{TE}} \right)^{-1} \left(r_{23}^{\text{TE}} \right)^{-1} e^{2\kappa_3 a} - 1, \tag{4.4}$$

in terms of the reflection coefficients for a single dielectric interface,

$$r_{ij}^{\text{TE}} = \frac{\kappa_i - \kappa_j}{\kappa_i + \kappa_j}. \tag{4.5}$$

Here $\kappa_i^2 = k^2 + \varepsilon_i \zeta^2$. For the TM mode, one merely replaces $\kappa_i \to \kappa_i/\varepsilon_i$ in the reflection coefficients.

In these papers, Lifshitz and collaborators showed that when the dielectric constants of the media became large, the result of Casimir for perfect conductors was recovered. They further showed that when the media were tenuous, so when

$$\varepsilon - 1 = 4\pi N\alpha \ll 1, \tag{4.6}$$

where N is the number density of polarizable molecules in the media, the Lifshitz force could be understood as the pairwise summation of Casimir-Polder forces between the molecules, and thereby rederived Eq. (3.2a). Many years later, these results and more were rederived by Schwinger and his postdocs.

49. "Casimir Effect in Dielectrics," J. Schwinger, L. L. DeRaad, Jr., and K. A. Milton, Ann. Phys. (N.Y.) **115**, 1–23 (1978). (A)

In this paper, as in the earlier Lifshitz papers, temperature dependence was also discussed. However, in order to recover the expected result for perfectly conducting plates, a prescription was adopted asserting that formally the limit $\varepsilon \to \infty$ had to be imposed before taking the zero frequency limit, which arises in the Matsubara sum over discrete frequencies that occurs for finite temperature T. That is, to pass from the zero-temperature Lifshitz formula to the finite-temperature expression, one replaces the frequency integral by a discrete sum:

$$\int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} f(\zeta) \to kT \sum_{m=-\infty}^{\infty} f(\zeta_m), \tag{4.7a}$$

where the Matsubara frequency is

$$\zeta_m = 2\pi m kT. \tag{4.7b}$$

(Here m is a positive or negative integer, including 0.) This prescription was designed to achieve agreement with the ideal metal limit, and guaranteed that the result was consistent with the third law of thermodynamics, that the entropy should vanish at zero temperature, although this was not widely appreciated at the time, and was not explicitly stated in the papers. But this prescription came under serious scrutiny at the dawn of the 21st century, and is now generally regarded as erroneous—see Sec. IX.

It should also be noted that there have been various attempts to divorce Casimir forces from the concept of zero-point energy. Notable among these are the following.

50. "Casimir Effect in Source Theory," J. Schwinger, Lett. Math. Phys. **1**, 43–47 (1975). (I)

51. "The Casimir Effect and the Quantum Vacuum," R.L. Jaffe, Phys. Rev. D **72**, 021301 (2005). (I)

See also Refs. [49] and [1]. However, it is the view of the author that since the predictions of the fluctuating field description built into quantum field theory are unaltered by these reformulations, there is no physical significance to these distinctions. The physical arena where these distinctions might be relevant is in connection with dark energy—See Sec. XVI.

There were many important papers on the Casimir effect published in the thirty-year period 1960–1990. Among these, special note should be given to the following.

52. "Vacuum Stress between Conducting Plates: An Image Solution," L. S. Brown and G. J. Maclay, Phys. Rev. **184**, 1272–1279 (1969). (I)

Brown and Maclay extracted the vacuum expectation value of the stress tensor that expresses both the stress (forces) on the plates as well as the local energy density,

$$\langle T^{\mu\nu}\rangle = u\left(4\hat{z}^{\mu}\hat{z}^{\nu} - g^{\mu\nu}\right), \tag{4.8}$$

where u is the energy density between the plates, separated by a distance a,

$$u = -\frac{\pi^2\hbar c}{720a^4}. \tag{4.9}$$

The stress tensor is expressed in terms of the metric tensor of flat space, $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, and where $\hat{z}^{\mu}$ is the unit vector in the z direction. Note that this says that the quantum energy density is constant between the plates (and zero outside). They further performed a full derivation of the Casimir energy at finite temperature between two parallel plates in the framework of thermal field theory, and observed a symmetry between low and high temperature behaviors, a type of duality.

V.   EXPERIMENT

Experimental confirmation of the Casimir force took a long time in coming. Early experiments in the 1950s and 1960s were relatively inconclusive, the best of which did "not contradict Casimir's theoretical prediction."

53. "Direct Measurement of the Molecular Attraction of Solid Bodies. I. Statement of the Problem and Measurement of the Force by Using Negative Feedback," B. V. Deriagin

(Derjaguin) and I. I. Abrikosova, Zh. Eksp. Teor. Fiz. **30** 993 (1956) [English transl.: Soviet Phys. JETP **3**, 819–829 (1957)]. (I)

54. "Measurements of Attractive Forces between Flat Plates," M. Sparnaay, Physica **24**, 751–764 (1958). (I)

The Lifshitz theory was convincingly confirmed by Sabisky and Anderson.

55. "Verification of the Lifshitz Theory of the van der Waals Potential Using Liquid-Helium Films," E. S. Sabisky and C. H. Anderson, Phys. Rev. A **7**, 790–806 (1973). (I)

They attributed the force of adhesion of a helium film to a substrate in a helium atmosphere in terms of the Lifshitz theory. The first convincing measurement of the Casimir force between conducting plates did not appear until 1997:

56. "Demonstration of the Casimir Force in the 0.6 to 6 μm Range," S. K. Lamoreaux, Phys. Rev. Lett. **78**, 5–8 (1997). (I)

This, and almost all subsequent Casimir measurements, involved not two parallel plates (which are exceedingly difficult to keep parallel), but a plane and a spherical lens (or a sphere), which force, when the separation distance between two objects is very small, could be deduced from that for parallel plates by the "proximity force approximation."

57. "Untersuchungen über die Reibung und Adhäsion, IV. Theorie des Anhaftens kleiner Teilchen," B. V. Derjaguin, Kolloid Z. **69**, 155–164 (1934). (I)

In this approximation, the force between a spherical conductor of radius R and a conducting plane separated by a distance d is

$$F = -\frac{\pi^3}{360}\frac{R}{d}\frac{\hbar c}{d^2}, \quad R \gg d. \tag{5.1}$$

It has proved impossible to reliably extend this approximation beyond the regime where the two objects are nearly touching. Only in recent years have exact and reliable numerical calculations become possible. (See Sec. XII.) Subsequent, more refined observations of the Casimir force include the following.

58. "Precision Measurement of the Casimir Force from 0.1 to 0.9 μm," U. Mohideen and A. Roy, Phys. Rev. Lett. **81**, 4549–4552 (1998). (A)

59. "Measurement of the Casimir Force between Dissimilar Metals," R. S. Decca, D. López, E. Fischbach, and D. E. Krause, Phys. Rev. Lett. **91**, 050402 (4 pages) (2003). (I)

60. "Precise Comparison of Theory and New Experiment for the Casimir Force Leads to Stronger Constraints on Thermal Quantum Effects and Long-Range Interactions," R. S. Decca, D. López, E. Fischbach, G. L. Klimchitskaya, D. E. Krause, and V. M. Mostepanenko, Ann. Phys. **318**, 37–80 (2005). (A)

61. "Tests of New Physics from Precise Measurements of the Casimir Pressure Between Two Gold-Coated Plates," R. S. Decca, D. López, E. Fischbach, G. L. Klimchitskaya, D. E. Krause, and V. M. Mostepanenko, Phys. Rev. D **75**, 077101 (4 pages) (2007). (A)

62. "Novel Constraints on Light Elementary Particles and Extra-Dimensional Physics from the Casimir Effect," R.S. Decca, D. López, E. Fischbach, G.L. Klimchitskaya, D.E. Krause, and V. M. Mostepanenko, Eur. Phys. J. C **51**, 963-975 (2007). (A)

63. "Measurement of the Casimir Force Between Parallel Metallic Surfaces," G. Bressi, G. Carugno, R. Onofrio, and G. Ruoso, Phys. Rev. Lett. **88**, 041804 (4 pages) (2002). (I)

The last reference is the only measurement of the Casimir force carried out between parallel surfaces, and consequently, the systematic errors are relatively large owing to the difficulty of maintaining parallelism. Electrostatic calibration is a problem in all these measurements, since the Casimir forces may be overwhelmed by electrostatic forces.

64. "Progress in Experimental Measurements of the Surface-Surface Casimir Force: Electrostatic Calibrations and Limitations to Accuracy," S. K. Lamoreaux, arXiv:1008.3640, invited review paper to appear in a Lecture Notes in Physics Volume on Casimir physics edited by Diego Dalvit, Peter Milonni, David Roberts, and Felipe da Rosa. (I)

The Casimir-Polder force between an atom and a conducting plate until recently had only been convincingly observed in a single experiment by Hinds' group:

65. "Measurement of the Casimir-Polder force," C. I. Sukenik, M. G. Boshier, D. Cho, V. Sundoghar, and E. A. Hinds, Phys. Rev. Lett. **70**, 560–563 (1993). (I)

This experiment actually measured the force between an atom and a pair of plates in a wedge configuration. This was compared to the force for an atom between parallel plates, which had been worked out by Barton:

66. "Quantum-Electrodynamic Level Shifts between Parallel Mirrors: Applications, Mainly to Rydberg States," G. Barton, Proc. Roy. Soc. London **410**, 175–200 (1987). (A)

The corresponding force on an atom by a perfectly conducting wedge was was only computed a decade later.

67. "Casimir-Polder Effect for a Perfectly Conducting Wedge," I. Brevik, M. Lygren, and V. N. Marachevsky, Ann. Phys. (N.Y.) **267**, 134–142 (1998). (I)

An earlier experiment by Hinds' group had seen the expected scaling of the van der Waals deflection of high n Rydberg atoms:

68. "Measuring the van der Waals Forces between a Rydberg Atom and a Metallic Surface," A. Anderson, S. Haroche, E. A. Hinds, W. Jhe, and D. Meschede, Phys. Rev. A **37**, 3594–3597 (1988). (I)

There were precursors to these investigations, for example,

69. "Interaction between a Neutral Atomic or Molecular Beam and a Conducting Surface," D. Raskin and P. Kusch, Phys. Rev. **179**, 712–721 (1969). (I)

More recently, however, the force between a Bose-Einstein condensate of ^{87}Rb atoms and a dielectric substrate was measured, even at different temperatures, by Eric Cornell's group:

70. "Measurement of the Casimir-Polder Force Through Center-of-Mass Oscillations of a Bose-Einstein Condensate," D. M. Harber, J. M. Obrecht, J. M. McGuirk, and E. A. Cornell, Phys. Rev. A **72**, 033610 (6 pages) (2005). (A)

71. "Measurement of the Temperature Dependence of the Casimir-Polder Force," J. M. Obrecht, R. J. Wild, M. Antezza, L. P. Pitaevskii, S. Stringari, and E. A. Cornell, Phys. Rev. Lett. **98**, 063201 (4 pages) (2007). (I)

VI.   DEVELOPMENTS IN VAN DER WAALS–CASIMIR-POLDER FORCES

Van der Waals forces can be thought of as arising from two-photon exchange:

72. "The Dispersion Theory of Dispersion Forces," G. Feinberg, J. Sucher, and C. -K. Au, Phys. Rept. **180**, 83–157 (1989). (I)

In this paper, the authors rederive the Casimir-Polder (3.2a) and London forces (3.1), and do so using a scattering matrix formalism, which has recently become popular (see Sec. XII). One-photon exchange processes yield the Coulomb interaction, while two-photon processes lead to a potential between two atoms, labeled A and B, of the form

$$V_{\text{CP}} = -\frac{D}{r^7}\hbar c, \tag{6.1}$$

where in general (for isotropic atoms)

$$D = \frac{23}{4\pi}\left(\alpha_E^A\alpha_E^B + \alpha_M^A\alpha_M^B\right) - \frac{7}{4\pi}\left(\alpha_E^A\alpha_M^B + \alpha_M^A\alpha_E^B\right), \tag{6.2}$$

in terms of the electric and magnetic polarizabilities $\alpha_{E,M}$. This result was first obtained by Feinberg and Sucher twenty years earlier, using dispersion theory.

73. "General Form of the Retarded van der Waals Potential," G. Feinberg and J. Sucher, J. Chem. Phys. **48**, 3333–3334 (1968). (I)

It was then reproduced on a more field theoretic basis by Boyer.

74. "Recalculations of Long-Range van der Waals Potentials," T. H. Boyer, Phys. Rev. **180**, 19–24 (1969). (I)

In their earlier paper Feinberg and Sucher noted that while the magnetic polarizability is usually negligible, it is not true for the hydrogen atom, where the coefficient of the $1/r^7$ is about 750 times that given by Casimir and Polder, although in that case the retarded interaction is only valid for $r > 21$ cm, making "the matter somewhat academic." This result (6.2) reduces to the Casimir-Polder result if $\alpha_M = 0$. In general, this allows for repulsive interactions, although such a situation is hard to realize in practice (see Sec. XIII). Feinberg and Sucher also note in their 1968 paper that "it would be very interesting, and a new confirmation of quantum electrodynamics, if some way could be found to detect such a repulsion." This is an equally relevant comment today. See also

75. "Quantum Zero-Point Energy and Long-Range Forces," T. H. Boyer, Ann. Phys. **56**, 474–503 (1970). (I)

76. "Simple Derivation of the Asymptotic Casimir Interaction of a Pair of Finite Systems," L. Spruch, J. F. Babb, and F. Zhou, Phys. Rev. A **49**, 2476–2482 (1994). (I)

77. "Long Range Electromagnetic Effects Involving Neutral Systems and Effective Field Theory," B. R. Holstein, Phys. Rev. D **78**, 013001 (11 pages) (2008). (I)

There are a great many developments in computing CP forces for atomic systems. For example, see:

78. "Casimir-Polder Forces on Excited Atoms in the Strong Atom-Field Coupling Regime," S. Y. Buhmann and D.-G. Welsch, Phys. Rev. A **77**, 012110 (16 pages) (2008). (A)

The analogous retarded Casimir interaction between an electron and a dielectric wall was given in the following.

79. "Retarded Casimir Interaction in the Asymptotic Domain of an Electron and a Dielectric Wall," Y. Tikochinsky and L. Spruch, Phys. Rev. A **48**, 4223–4235 (1993). (I)

VII. FIELD THEORY OF CASIMIR FORCES AND QCD

Symanzik thirty years ago provided a general renormalized field-theoretic basis to the Casimir effect:

80. "Schrödinger Representation and Casimir Effect in Renormalizable Quantum Field Theory," K. Symanzik, Nucl. Phys. B **190**, 1–44 (1981). (A)

He noted there, perhaps before it was generally recognized, that "the Casimir potential between disjoint surfaces is always well defined. That for a single surface, e.g. a sphere, in general is not." There are still issues to be resolved in Casimir self-energies.

81. "Local and Global Casimir Energies: Divergences, Renormalization, and the Coupling to Gravity," K. A. Milton, arXiv:1005.0031 (53 pages), invited review paper to appear

in a Lecture Notes in Physics Volume on Casimir physics edited by Diego Dalvit, Peter Milonni, David Roberts, and Felipe da Rosa. (A)

See Sec. XIV.

The gauge theory of strong interactions, quantum chromodynamics (QCD), has still not been completely solved. A simple model that encompasses some of the expected features of the full theory is the MIT bag model. In this model, there is an important parameter that is supposed to represent the effects of zero-point fluctuations of the quark and gluon fields within the confining bag. Crude estimates of these effects were computed some thirty years ago:

82. "Zero-point Energy in Bag Models," K. A. Milton, Phys. Rev. D **22**, 1441–1443 (1980). (I)

83. "Zero-point Energy of Confined Fermions," K. A. Milton, Phys. Rev. D **22**, 1444–1451 (1980). (A)

Other parameters of QCD, of phenomenological significance, could also be estimated in a bag-model context:

84. "Quark and Gluon Condensates in a Bag Model of the Vacuum," K. A. Milton, Phys. Lett. B **104**, 49–54 (1981). (A)

In this connection, the important early paper of Bender and Hays must be mentioned:

85. "Zero-Point Energy of Fields in a Finite Volume," C. M. Bender and P. Hays, Phys. Rev. D **14**, 2622 (1976). (I)

About the same time, Feinberg and Sucher suggested that there might be a strong van der Waals force between hadrons, which in turn could be a residual effect of QCD within the hadrons:

86. "Is There a Strong van der Waals Force Between Hadrons?" G. Feinberg, J. Sucher, Phys. Rev. D **20**, 1717–1735 (1979). (I)

The authors noted that such long-range forces are subject to severe phenomenological constaints, and for example are absent in the bag model. See also the following:

87. "A Multipole Expansion and the Casimir-Polder Effect in Quantum Chromodynamics," G. Bhanot, W. Fischler, and S. Rudaz, Nucl. Phys. B **155**, 208–236 (1979).

Casimir-Polder forces in QCD were considered more recently in

88. "Long-Range Forces of QCD," H. Fujii and D. Kharzeev, Phys. Rev. D **60**, 114039 (12 pages) (1999). (A)

An analysis of Casimir energies relevant to both QCD and cosmology was given in the important paper by Blau, Visser, and Wipf:

89. "Zeta Functions and the Casimir Energy," S. K. Blau, M. Visser, and A. Wipf, Nucl. Phys. B **310**, 163–180 (1988). (A)

VIII.   EXACT CASIMIR FORCE FROM C-P FORCES

We can reverse the derivation of the Casimir-Polder force from the Lifshitz forces between dilute dielectrics referred to above, in Sec. IV, to calculate quantum vacuum forces between macroscopic bodies made up of polarizable molecules. In many cases these forces can be given closed form through summing the Casimir-Polder potentials (3.2a):

90. "Exact Results for Casimir Interactions between Dielectric Bodies: The Weak-Coupling or van der Waals Limit," K. A. Milton, P. Parashar, and J. Wagner, Phys. Rev. Lett. **101**, 160402 (4 pages) (2008). (I)

For example, one can calculate exact forces between dilute bodies of various shapes, and can exhibit a Casimir torque between a semi-infinite dielectric slab and a dielectric rectangular block: For a fixed distance between the center of mass of the block and the slab, the equilibrium configuration of the block ranges from that in which the shortest side faces the plane, for large distances, to one in which the center of mass of the block is directly above the point of contact when the bodies just touch. This is a "tidal" effect hinging on the fact that the CP force falls off with distance.

Such pairwise summations of CP forces were first carried out to calculate self energies. For example, the self energy of a ball of radius a made of a dilute dielectric $|\varepsilon - 1| \ll 1$, was evaluated by summing Casimir-Polder energies.

91. "Observability of the Bulk Casimir Effect: Can the Dynamical Casimir Effect be Relevant to Sonoluminescence?" K. A. Milton and Y. J. Ng, Phys. Rev. E **57**, 5504–5510 (1998). (I)

The resulting self-energy was found to be

$$E_{\text{dilute sphere}} = \frac{23}{2536\pi a}(\varepsilon - 1)^2, \tag{8.1}$$

which was later verified by a full field-theoretic calculation.

92. "Identity of the van der Waals Force and the Casimir Effect and the Irrelevance of These Phenomena to Sonoluminescence," I. Brevik, V. N. Marachevsky, and K. A. Milton, Phys. Rev. Lett. **82**, 3948–3951 (1999). (A)

The corresponding energy for a dilute dielectric circular cylinder is zero, verified both by summing Casimir-Polder energies, and a full calculation of the Casimir energy:

93. "Mode-by-Mode Summation for the Zero Point Electromagnetic Energy of an Infinite Cylinder," K. A. Milton, A. V. Nesterenko, and V. V. Nesterenko, Phys. Rev. D **59**, 105009 (9 pages) (1999). (I)

94. "Casimir Energy for a Dielectric Cylinder," I. Cavero-Peláez and K. A. Milton, Ann. Phys. (N.Y.) **320**, 108–134 (2005). (A)

95. "Casimir Energy for a Purely Dielectric Cylinder by the Mode Summation Method," A. Romeo and K. A. Milton, Phys. Lett. B **621**, 309–317 (2005). (A)

Vanishing of weakly coupled cylindrical Casimir energies is a universal feature, which is not yet well understood.

Actually, there is a much more extensive literature calculating Casimir forces between dilute bodies based on the nonretarded van der Waals interaction going like r^{-6}. Much of this is summarized in the book by Parsegian.

96. Van der Waals Forces: A Handbook for Biologists, Chemists, Engineers, and Physicists, V. Adrian Parsegian (Cambridge University Press, Cambridge, 2007). (I)

Nonretarded van der Waals forces have importance in biological systems, but the use of the unretarded form of the force is typically restricted to distances below 10 nm. Special mention should be made of the following paper.

97. "On the Macroscopic Theory of van der Waals Forces," N. G. Van Kampen, B. R. A. Nijboer, and K. Schram, Phys. Lett. A **26**, 307–308 (1968). (I)

They rederived the Lifshitz formula in the non-retarded regime; we note the following precusor and follow-up:

98. "Microscopic Derivation of Macroscopic van der Waals Forces," M. J. Renne and B. R. A. Nijboer, Chem. Phys. Lett. **1**, 317–320 (1967). (I)

99. "On the Macroscopic Theory of Retarded van der Waals Forces," K. Schram, Phys. Lett. A **43**, 282–284 (1973). (I)

Broad strokes of how Casimir phenomena could be relevant in chemistry are given in the following.

100. "Casimir Chemistry," D. P. Sheehan, J. Chem. Phys. **131**, 104706 (11 pages) (2009). (E)

IX. THEORETICAL CONTROVERSIES AND EXPERIMENTAL CONSTRAINTS

Some years ago Bordag suggested that the theory of Casimir-Polder forces might be subject to modification, if the conducting boundaries are "thin." The resulting force is modified from that of the standard theory by about 13%, which is consistent with the experimental error in the Sukenik experiment, Ref. [65].

101. "Reconsidering the Quantization of Electrodynamics with Boundary Conditions and Some Measurable Consequences," M. Bordag, Phys. Rev. D **70**, 085010 (11 pages) (2004). (I)

This result remains controversial, since no sign of this effect appears in the calculation based on field-strength tensors found in the Schwinger-DeRaad-Milton paper, Ref. [49].

A much more heated controversy has arisen concerning the temperature dependence of the Casimir effect for metals. As mentioned above, in Sec. IV, the prescription of extracting the temperature dependence for perfect metals used by Lifshitz, made most explicit in Ref. [49], was justified later by a desire not to conflict with the third law of thermodynamics. But Boström and Sernelius argued that this could not be correct.

102. "Thermal Effects on the Casimir Force in the 0.1–5 μm Range," M. Boström and Bo E. Sernelius, Phys. Rev. Lett. **84**, 4757–4760 (2000). (I)

They showed that the TE zero-frequency mode should be excluded for metals, just as it is for dielectrics. This has led to some intense discussion, partly to do with the thermodynamic inconsistency.

103. "Correlation of Energy and Free Energy for the Thermal Casimir Force between Real Metals." V. B. Bezerra, G. L. Klimchitskaya, and V. M. Mostepanenko, Phys. Rev. A **66**, 062112 (13 pages) (2002). (A)

104. "Violation of the Nernst Heat Theorem in the Theory of the Thermal Casimir Force between Drude Metals," V. B. Bezerra, G. L. Klimchitskaya, V. M. Mostepanenko, and C. Romero, Phys. Rev. A **69**, 022119 (9 pages) (2004). (A)

In fact, it has now been shown that for real metals (not fictional ideal perfect crystal lattices) the third law is satisfied, in that the entropy vanishes at zero temperature:

105. "Analytical and Numerical Verification of the Nernst Theorem for Metals," J. S. Høye, I. Brevik, S. A. Ellingsen, and J. B. Aarseth, Phys. Rev. E **75**, 051127 (8 pages) (2007). (A)

106. "Analytical and Numerical Demonstration of How the Drude Dispersive Model Satisfies Nernst's Theorem for the Casimir Entropy," I. Brevik, S. A. Ellingsen, J. S. Høye, and K. A. Milton, J. Phys. A: Math. Theor. **41**, 164017 (10 pages) (2008). (I)

Now there is general agreement that the modified theory is correct, although the authors of Refs. [103] and [104] would disagree. This means that at low temperature there is a linear temperature correction term, and at high temperature the usual linear temperature

dependence is reduced by a factor of 2. If the metal is otherwise ideal, the pressure given by the modified theory is [$\zeta(3) = 1.20206$ is the Riemann zeta function at 3]

$$P = -\frac{\pi^2 \hbar c}{240 a^4}\left[1 + \frac{16}{3}\left(\frac{akT}{\hbar c}\right)^4\right] + \frac{\zeta(3)}{8\pi a^3}kT, \quad kTa/\hbar c \ll 1, \tag{9.1a}$$

$$P = -\frac{\zeta(3)}{8\pi a^3}kT, \quad kTa/\hbar c \gg 1, \tag{9.1b}$$

while the linear term in the low-temperature limit is not present in the conventional theory, and the linear term in the high-temperature limit is twice as large in the conventional theory. The reduction of the high-temperature effect seen in Eq. (9.1b) was found in independent theoretical approaches:

107. "Casimir Force between Two Ideal-Conductor Walls Revisited," B. Jancovici and L. Šamaj, Europhys. Lett. **72**, 35–41 (2005). (I)

108. "The Casimir Force at High Temperature," P. R. Buenzli and Ph. A. Martin, Europhys. Lett. **72**, 42–48 (2005). (I)

The thermodynamic problem associated with Eq. (9.1a) is that the corresponding free energy is

$$F \approx F_0 + \frac{\zeta(3)}{16\pi a^2}kT, \quad P = -\left(\frac{\partial F}{\partial a}\right)_T, \tag{9.2}$$

at low temperature, so the entropy/area does not vanish at low T:

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = -\frac{\zeta(3)}{16\pi a^2}k. \tag{9.3}$$

But, in fact, it turns out that, because for real metals, $\zeta^2 \varepsilon(\zeta) \to 0$ as the frequency $\zeta \to 0$, for very low temperatures the free energy behaves not as Eq. (9.2) but as

$$F = F_0 + \gamma T^2, \quad kT \ll 20\text{mK}, \tag{9.4}$$

where γ is a constant depending on the resistivity of the metal. [$\gamma(T = 0) \neq 0$ for a real metal; $\gamma(T = 0) = 0$ only for a perfect ideal lattice metal.] Thus the entropy vanishes at $T = 0$, and there is no thermodynamic inconsistency. These results agree fairly closely with those obtained through exact integration of the Lifshitz formula using optical data for real metals, such as gold. The prediction (9.1a) is claimed by the authors of Refs. [60], [61], [62] to be insconsistent with their experiments, but this is not persuasive to many observers,

since the thermal effect at room temperatures for experiments conducted at the 100 nm distance range is only a few percent. The prediction (9.1b) has now been verified by a new experiment by Lamoreaux's group in the micrometer range:

109. "Observation of the Thermal Casimir Force," A. O. Sushkov, W. J. Kim, D. A. R. Dalvit, and S. K. Lamoreaux, Nature Physics **7**, 230–233 (2011). (I)

See also the commentary on this paper:

110. "The Casimir Force: Feeling the Heat," K. A. Milton, Nature Physics **7**, 190–191 (2011). (E)

Not surprisingly, Mostepanenko and collaborators do not accept this result, and argue that surface imperfections in the large spheres used in this experiment render the experiment inconclusive.

111. "Impact of Surface Imperfections on the Casimir Force for Lenses of Centimeter-Size Curvature Radii," V. B. Bezerra, G. L. Klimchitskaya, U. Mohideen, V. M. Mostepanenko, and C. Romero, Phys. Rev. B **83**, 075417 (13 pages) (2011). (I)

There is also a controversy about the thermal Casimir force between semiconductors. In this case, the discontinuity involves the zero-frequency TM mode. The problem was identified in 2005.

112. "Thermal Quantum Field Theory and the Casimir Interaction between Dielectrics," B. Geyer, G. L. Klimchitskaya, and V. M. Mostepanenko, Phys. Rev. D **72**, 085009 (20 pages) (2005). (A)

The difficulty was elaborated in further papers by the same authors. A different perspective was given by in the following reference.

113. "Temperature Correction to Casimir-Lifshitz Free Energy at Low Temperatures: Semiconductors," S. Å. Ellingsen, I. Brevik, J. S. Høye, and K. A. Milton, Phys. Rev. E **78**, 021117 (11 pages) (2008). (A)

A related experimental anomaly was discovered when the carrier concentration of a semiconductor was changed by illumination by a laser. The data indicated that the conductivity must be excluded for a low carrier concentration, but included in the case of high carrier

concentration. Inclusion of the conductivity (which is nonzero in all cases) in both situations led to disagreement with the experimental results.

114. "Control of the Casimir Force by the Modification of Dielectric Properties with Light," F. Chen, G. L. Klimchitskaya, V. M. Mostepanenko, and U. Mohideen, Phys. Rev. B **76**, 035338 (15 pages) (2007). (I)

These issues have yet to be satisfactorily resolved.

On the other hand, there seems to be no controversy about the thermal Casimir-Polder force between an atom and a metallic surface. However, there appears to be a conflict between theory and experiment, and with the Nernst heat theorem for the CP force between an atom and a dielectric if the dc conductivity of the latter in included.

115. "Conductivity of Dielectric and Thermal Atom-Wall Interaction," G. L. Klimchitskaya and V. M. Mostepanenko, J. Phys. A: Math. Theor. **41**, 312002 (8 pages) (2008). (I)

116. "Thermal Casimir-Polder Force between an Atom and a Dielectric Plate: Thermodynamics and Experiment," G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, J. Phys. A: Math. Theor. **41**, 432001 (9 pages) (2009). (I)

As noted above, this problem of the effect of small conductivity is still under investigation.

117. "Thermal Lifshitz Force between an Atom and a Conductor with a Small Density of Carriers," L. P. Pitaevskii, Phys. Rev. Lett. **101**, 163202 (4 pages) (2008). (I)

Interestingly, the CP potential may be temperature-independent even when the background contains a large number of thermal photons.

118. "Temperature-Independent Casimir-Polder Forces Despite Large Thermal Photon Numbers," S. Å. Ellingsen, S. Y. Buhmann, and S. Scheel, Phys. Rev. Lett. **104**, 223003 (4 pages) (2010). (I)

The classical regime of linear temperature dependence (3.2b) is never reached. There is also recent interesting work on out-of-equilibrium interatomic forces:

119. "Casimir-Lifshitz Force out of Thermal Equilibrium," M. Antezza, L. P. Pitaevskii, S. Stringari, and V. B. Svetovoy, Phys. Rev. A **77**, 022901 (22 pages) (2008). (A)

120. "Nonequilibrium Forces between Neutral Atoms Mediated by a Quantum Field," R. O. Behunin and B.-L. Hu, Phys. Rev. A **82**, 022507 (12 pages) (2010). (I)

121. "Dynamics of Thermal Casimir-Polder Forces on Polar Molecules," S. Å. Ellingsen, S. Y. Buhmann, and S. Scheel, Phys. Rev. A **79**, 052903 (6 pages) (2009). (I)

The latter reference shows that out of equilibrium, nonmonotonic (repulsive) forces can occur.

X.   NANOTECHNOLOGICAL APPLICATIONS

The first experimental papers that demonstrated that quantum vacuum forces could be useful in micromachinery appeared ten years ago:

122. "Stiction, Adhesion Energy, and the Casimir Effect in Micromechanical Systems," E. Buks and M. L. Roukes, Phys. Rev. B **63**, 033402 (4 pages) (2001). (I)

123. "Metastability and the Casimir Effect in Micromechanical Systems," E. Buks and M. L. Roukes, Europhys. Lett. **54**, 220–226 (2001). (I)

124. "Quantum Mechanical Actuation of Microelectromechanical Systems by the Casimir Force," H. B. Chan, V. A. Aksyuk, R. N. Kleiman, D. J. Bishop, and F. Capasso, Science **291**, 1941–1944 (2001). (I)

125. "Nonlinear Micromechanical Casimir Oscillator," H. B. Chan, V. A. Aksyuk, R. N. Kleiman, D. J. Bishop, and F. Capasso, Phys. Rev. Lett. **87**, 211801 (4 pages) (2001). (I)

The first two references here find forces that do not agree with the simple Casimir theory.

There have been many theoretical papers proposing micro-machinery and nano-machinery actuated by Casimir-type forces, for example, the proposal for a noncontact rack and pinion:

126. "Noncontact Rack and Pinion Powered by the Lateral Casimir Force," A. Ashourvan, M. Miri, and R. Golestanian, Phys. Rev. Lett. **98**, 140801 (4 pages) (2007). (I)

The following papers suggested the development of non-contact gears:

127. "Exploring the Quantum Vacuum with Cylinders," F. C. Lombardo, F. D. Mazzitelli, and P. I. Villar, J. Phys. A: Math. Theor. **41**, 164009 (10 pages) (2008). (I)

128. "Noncontact Gears. II. Casimir Torque Between Concentric Corrugated Cylinders for the Scalar Case," I. Cavero-Peláez, K. A. Milton, P. Parashar, and K. V. Shajesh, Phys. Rev. D **78**, 065018 (7 pages) (2008). (A)

For a perspective from the Paris group, see the following.

129. "The Casimir Effect in the Nanoworld," C. Genet, A. Lambrecht, and S. Reynaud, Eur. Phys. J. Special Topics **160**, 183 (8 pages) (2008) [arXiv:0809.0254]. (I)

XI.   CASIMIR FRICTION

Lateral as opposed to normal forces between plates have been considered in a variety of contexts in recent years. For example, if the plates are corrugated, there should be a lateral (sideways) force attempting to bring the peaks of the lower surface closest to the troughs in the upper surface. This was first observed by Mohideen's group:

130. "Demonstration of the Lateral Casimir Force," F. Chen, U. Mohideen, G. L. Klimchitskaya, and V. M. Mostepanenko, Phys. Rev. Lett. **88**, 101801 (4 pages) (2002). (I)

Comparison with the best theory at the time, the proximity-force approximation, was not very satisfactory:

131. "Experimental and Theoretical Investigation of the Lateral Casimir Force Between Corrugated Surfaces," F. Chen, U. Mohideen, G. L. Klimchitskaya, and V. M. Mostepanenko, Phys. Rev. A **66**, 032113 (11 pages) (2002). (I)

132. "Lateral Casimir Force Beyond the Proximity Force Approximation: A Nontrivial Interplay Between Geometry and Quantum Vacuum," R. B. Rodrigues, P. A. Maia Neto, A. Lambrecht, and S. Reynaud, Phys. Rev. A **75**, 062108 (10 pages) (2007). (I)

The definitive observation, and comparison with the multiple-scattering theory, was given in 2009:

133. "Lateral Casimir Force Between Sinusoidally Corrugated Surfaces: Asymmetric Profiles, Deviations from the Proximity Force Approximation, and Comparison with Exact Theory," H.-C. Chiu, G. L. Klimchitskaya, V. N. Marachevsky, V. M. Mostepanenko, and U. Mohideen, Phys. Rev. B **81**, 115417 (20 pages)(2010). (I)

This is relevant to nanotechnological applications. (See Sec. X.) However, with deeper trenches on the surfaces, the deviation from the proximity force approximation is more pronounced.

134. "Measurement of the Casimir Force between a Gold Sphere and a Silicon Surface with Nanoscale Trench Arrays," H. B. Chan, Y. Bao, J. Zou, R. A. Cirelli, F. Klemens, W. M. Mansfield, and C. S. Pai, Phys. Rev. Lett. **101**, 030401 (4 pages) (2008). (I)

The results agree with theoretical calculations:

135. "Casimir Interaction of Dielectric Gratings," A. Lambrecht and V. N. Marachevsky, Phys. Rev. Lett. **101**, 160403 (4 pages) (2008). (I)

But can there be a transverse force in the absence of corrugations? Sometime ago, researchers started considering the transverse force between plane surfaces in relative parallel motion:

136. "On the Contribution of Macroscopic van der Waals Interactions to Frictional Force," E. V. Teodorovich, Proc. R. Soc. London, Ser. A, **362**, 71–77 (1978). (I)

137. "Van der Waals' Friction," L. S. Levitov, Europhys. Lett. **8**, 499–504 (1989). (I)

138. "The 'Friction' of Vacuum, and Other Fluctuation-Induced Forces," M. Kardar and R. Golestanian, Rev. Mod. Phys. **71**, 1233–1245 (1999). (I)

Some authors find no friction at all:

139. "No Quantum Friction Between Uniformly Moving Plates," T. G. Philbin and U. Leonhardt, New J. Phys. **11**, 033035 (18 pages) (2009). (I)

A useful overview is

140. "Quantum Friction—Fact or Fiction?" J. B. Pendry, New J. Phys. **12**, 033028 (7 pages) (2010). (I)

The extensive work of Dedkov and Kyasov should be cited:

141. "Fluctuation Electromagnetic Slowing Down and Heating of a Small Neutral Particle Moving in the Field of Equilibrium Background Radiation," G. V. Dedkov and A. A. Kyasov, Phys. Lett. A **339** 212–216 (2005). (I)

See also earlier papers cited therein, as well as the following.

142. "Casimir-Polder Forces on Moving Atoms," S. Scheel and S. Y. Buhmann, Phys. Rev. A **80**, 042902 (11 pages) (2009). (A)

A statistical-mechanical treatment of the Casimir friction problem, based on dielectric particles moving with constant relative velocity, is given in

143. "Casimir Friction Force and Energy Dissipation for Moving Harmonic Oscillators," J. S. Høye and I. Brevik, Europhys. Lett. **91**, 60003 (5 pages) (2010). (I)

They conclude that the friction vanishes at zero temperature, but is nonzero at nonzero temperature. A different perspective on this is given by Barton:

144. "On van der Waals Friction: I. Between Two Atoms," G. Barton, New J. Phys. **12**, 113044 (10 pages) (2010). (I)

145. "On van der Waals Friction. II: Between Atom and Half-Space," G. Barton, New J. Phys. **12**, 113045 (13 pages) (2010). (I)

For example, Barton finds friction at all temperatures, a difference with the result found in Ref. [143] that he attributes to a different choice of initial conditions. However, the authors of that reference demonstrate that both approaches are in fact equivalent.

146. "Casimir Friction in Terms of Moving Harmonic Oscillators: Equivalence Between Two Different Formulations," J. S. Høye and I. Brevik, arXiv:1101.1241. (I)

XII. RECENT THEORETICAL DEVELOPMENTS

In the last few years, great progress has been made in calculating forces between objects of arbitrary shape. Previously, deviations of forces from that of plane surfaces had been calculated using the proximity force approximation [57]. The associated errors of this

approximation were unknown. One approach for transcending such limitations was the "world-line" method developed by Gies and collaborators:

147. "Worldline Algorithms for Casimir Configurations," H. Gies and K. Klingmuller, Phys. Rev. D **74**, 045002 (12 pages) (2006). (A)

148. "Geothermal Casimir Phenomena," K. Klingmuller and H. Gies, J. Phys. A **41**, 164042 (8 pages) (2008). (A)

149. "Interplay between Geometry and Temperature for Inclined Casimir Plates." A. Weber and H. Gies, Phys. Rev. D **80**, 065033 (17 pages) (2009). (A)

150. "Nonmonotonic Thermal Casimir Force from Geometry-Temperature Interplay," A. Weber and H. Gies, Phys. Rev. Lett. **105**, 040403 (4 pages) (2010). (A)

151. "Geometry-Temperature Interplay in the Casimir Effect," H. Gies and A. Weber, Int. J. Mod. Phys. A **25**, 2279–2292 (2010). (A)

Unfortunately, no one has yet succeeded in applying this method to other than perfect Dirichlet boundary conditions.

Much of the recent advance has to do with the application of multiple scattering techniques, which of course have a long history. These developments early on included cylindrical calculations, which might be amenable to experimentation.

152. "Exact Zero-Point Interaction Energy Between Cylinders," F. D. Mazzitelli, D. A. R. Dalvit, and F. C. Lombardo, New J. Phys. **8**, 240 (21 pages) (2006). (A)

153. "Exact Casimir Interaction between Eccentric Cylinders," D. A. R. Dalvit, F. C. Lombardo, F. D. Mazzitelli, and R. Onofrio, Phys. Rev. A **74**, 020101 (4 pages) (2006). (A)

More general developments appeared in the following papers.

154. "Scalar Casimir Effect between Dirichlet Spheres or a Plate and a Sphere," A. Bulgac, P. Magierski, and A. Wirzba, Phys. Rev. D **73**, 025007 (14 pages) (2006). (A)

155. "Casimir Interaction between a Plate and a Cylinder," T. Emig, R. L. Jaffe, M. Kardar, and A. Scardicchio, Phys. Rev. Lett. **96**, 08040 (4 pages) (2006). (A)

156. "Opposites Attract—A Theorem about the Casimir Force," O. Kenneth and I. Klich, Phys. Rev. Lett. **97**, 160401 (4 pages) (2006). (I)

157. "Fluctuation-Induced Quantum Interactions between Compact Objects and a Plane Mirror," T. Emig, J. Stat. Mech.–Th. Exp. P04007 (33 pages) (2008). (A)

In particular, the sphere-plate configuration used in experiments, was analyzed in the last reference and in the following.

158. "Scattering Theory Approach to Electrodynamic Casimir Forces," S. J. Rahi, T. Emig, N. Graham, R. L. Jaffe, and M. Kardar, Phys. Rev. D **80**, 085021 (27 pages) (2009). (A)

159. "Casimir Energy between a Plane and a Sphere in Electromagnetic Vacuum," P. A. Maia Neto, A. Lambrecht, and S. Reynaud, Phys. Rev. A **78**, 012115 (4 pages) (2008). (A)

Work continues on using such methods to extract analytic corrections to the proximity force approximation:

160. "Casimir Force for a Sphere in Front of a Plane beyond Proximity Force Approximation," M. Bordag and V. Nikolaev, J. Phys. A: Math. Gen. **41**, 164002 (10 pages) (2008). (A)

161. "First Analytic Correction Beyond the Proximity Force Approximation in the Casimir Effect for the Electromagnetic Field in Sphere-Plane Geometry," M. Bordag and V. Nikolaev, Phys. Rev. D **81**, 065011 (14 pages) (2010). (A)

Although numerical and analytic methods agree for Dirichlet boundary conditions, discrepancies persist for Neumann and electromagnetic boundaries. See Refs. [157], [159], and the following.

162. "Numerical Evaluation of the Casimir Interaction between Cylinders," F. C. Lombardo, F. D. Mazzitelli, and P. I. Villar, Phys. Rev. D **78**, 085009 (11 pages) (2008). (A)

While the above references are typically concerned with rather small corrections that might be seen in the current generation of experiments, it was immediately recognized that much more general geometries were amenable to calculations:

163. "Casimir Forces between Arbitrary Compact Objects," T. Emig, N. Graham, R. L. Jaffe, and M. Kardar, Phys. Rev. Lett. **99**, 170403 (4 pages) (2007). (I)

164. "Multiple Scattering Methods in Casimir Calculations," K. A Milton and J. Wagner, J. Phys. A: Math. Theor. **41**, 155402 (24 pages) (2008). (A)

165. "Weak Coupling Casimir Energies for Finite Plate Configurations," J. Wagner, K. A. Milton, and P. Parashar, J. Phys.: Conf. Ser. **161**, 012022 (10 pages) (2009). (I)

Very interesting modelling of Casimir-Polder forces are provided by recent calculations of quantum vacuum forces and torques on a small object enclosed in a spherical conducting shell (or exterior to a conducting shell):

166. "Casimir Interactions of an Object Inside a Spherical Metal Shell," S. Zaheer, S. J. Rahi, T. Emig, and R. L. Jaffe, Phys. Rev. A **81**, 0305020(R) (4 pages) (2010). (I)

167. "Casimir Potential of a Compact Object Enclosed by a Spherical Cavity," S. Zaheer, S. J. Rahi, T. Emig, and R. L. Jaffe, Phys. Rev. A **82**, 052507 (10 pages) (2010). (I)

Simultaneously with the development of these analytical techniques, numerical methods, based on finite-difference solutions of Maxwell's equations, have been developed by an independent MIT group:

168. "Efficient Computation of Casimir Interactions between Arbitrary 3D Objects," M. T. Homer Reid, A. W. Rodriguez, J. White, and S. G. Johnson, Phys. Rev. Lett. **103**, 040401 (4 pages) (2009). (I)

169. "Casimir Forces in the Time Domain II: Applications," A. P. McCauley, A. W. Rodriguez, J. D. Joannopoulos, and S. G. Johnson, Phys. Rev. A **81**, 012119 (10 pages) (2010). (A)

170. "Achieving a Strongly Temperature-Dependent Casimir Effect," A. W. Rodriguez, D. Woolf, A. P. McCauley, F. Capasso, J. D. Joannopoulos, and S. G. Johnson, Phys. Rev. Lett. **105**, 060401 (4 pages) (2010). (I)

XIII. REPULSIVE CASIMIR FORCES

Motivated by possible nanotechnological applications, there has been a resurgence of interest in repulsive Casimir and Casimir-Polder forces. Of course, it was known by Lifshitz and collaborators that repulsion could be achieved if two different dielectric materials, of permittivities ε_1 and ε_2 were separated by a medium with an intermediate value of the dielectric constant $\varepsilon_1 < \varepsilon_3 < \varepsilon_2$. This has recently be confirmed experimentally:

171. "Measured Long-Range Repulsive Casimir-Lifshitz Forces," J. N. Munday, F. Capasso, and V. A. Parsegian, Nature **457**, 170–173 (2009). (I)

Precursors of this experiment should also be mentioned.

172. "Direct Measurement of Repulsive van der Waals Interactions Using an Atomic Force Microscope," A. Milling, P. Mulvaney, and I. Larson, J. Colloid Interf. Sci. **180**, 460–465 (1996). (I)

173. "Direct Measurement of Repulsive and Attractive van der Waals Forces between Inorganic Materials," A. Meurk, P. F. Luckham, and L. Bergstrom, Langmuir **13**, 3896–3899 (1997). (I)

174. "Repulsive van der Waals Forces for Silica and Alumina," S. Lee and W. M. Sigmund, J. Colloid Interf. Sci. **243**, 365–369 (2001). (I)

175. "AFM Study of Repulsive van der Waals Forces between Teflon AF$^{\mathrm{TM}}$ Thin Film and Silica or Alumina," S. Lee and W. M. Sigmund, Colloids Surf. A **204**, 43–50 (2002). (I)

176. "Superlubricity Using Repulsive van der Waals Forces," A. A. Feiler, L. Bergström, and M. W. Rutland, Langmuir **24**, 2274–2276 (2008). (I)

More interestingly, Boyer demonstrated in 1968 that the Casimir self-stress on a sphere was repulsive, whereas that for a conducting circular cylindrical shell is attractive.

177. "Quantum Electromagnetic Zero-Point Energy of a Conducting Spherical Shell and the Casimir Model for a Charged Particle," T. H. Boyer, Phys. Rev. **174**, 1764–1776 (1968). (I)

178. "Electromagnetic Waves Near Perfect Conductors. II. Casimir Effect," R. Balian and B. Duplantier, Ann. Phys. (N.Y.) **112**, 165–208 (1978). (I)

179. "Casimir Self-Stress on a Perfectly Conducting Spherical Shell," K. A. Milton, L. L. DeRaad, Jr., and J. Schwinger, Ann. Phys. (N.Y.) **115**, 388–403 (1978). (I)

180. "Casimir Self-Stress on a Perfectly Conducting Cylindrical Shell," L. L. DeRaad, Jr. and K. A. Milton, Ann. Phys. (N.Y.) **136**, 229–242. (A)

Casimir-Polder forces could be repulsive if they involved magnetic moment couplings, rather than the usual electric-dipole couplings:

181. "Theory of Casimir-Polder Forces," B.-S. Skagerstam, P. K. Rekdal, and A. H. Vaskinn, Phys. Rev. A **80**, 022902 (12 pages) (2009). (A)

This is consistent with the result (6.2). In fact, many years ago Boyer showed that Casimir force between a perfect electrically conducting plate and perfect magnetically conducting plate (that is, the permittivity $\varepsilon \to \infty$, and the permeability $\mu \to \infty$, respectively) is repulsive.

182. "Van der Waals Forces and Zero-Point Energy for Dielectric and Permeable Materials." T. H. Boyer, Phys. Rev. A **9**, 2078–2084 (1974). (I)

This suggests that properly designed metamaterials might exhibit repulsive forces, but the severe difficulty is that the unusual magnetic properties must persist over a very wide frequency range to achieve an observable effect, since the reflection coefficients appear inside an integral.

183. "Casimir Force between Designed Materials: What Is Possible and What Not," C. Henkel and K. Joulain, Europhys. Lett. **72**, 929–935 (2005). (I)

184. "Casimir Repulsion and Metamaterials," I. G. Pirozhenko and A. Lambrecht, J. Phys. A 41, 164015 (8 pages) (2008). (A)

185. "Casimir-Lifshitz Theory and Metamaterials," F. S. S. Rosa, D. A. R. Dalvit, and P. W. Milonni, Phys. Rev. Lett. **100**, 183602 (4 pages) (2008). (I)

186. "First-Principles Study of Casimir Repulsion in Metamaterials," V. Yannopapas and N. V. Vitanov, Phys. Rev. Lett. **103**, 120401 (4 pages) (2009). (I)

187. " Physical Restrictions on the Casimir Interaction of Metal-Dielectric Metamaterials: An Effective-Medium Approach," M. G. Silveirinha and S. I. Maslovski, Phys. Rev. A **82**, 052508 (5 pages) (2010). (A)

188. "Microstructure Effects for Casimir Forces in Chiral Metamaterials," McCauley et al., Phys. Rev. B **82**, 165108 (5 pages) (2010). (I)

It also might be possible to achieve repulsion by focusing electromagnetic fields with parabolic mirrors:

189. "Focusing Vacuum Fluctuations. II," L. H. Ford and N. F. Svaiter, Phys. Rev. A **66**, 062106 (13 pages) (2002). (I)

Very recently, numerical calculations have shown that an elongated cylinder directly above a circular hole in a conducting sheet experiences a Casimir force that is attractive at large distances but is repulsive at distances that are of the order of the diameter of the hole. The equilibrium point is, of course, unstable, since the energy is lowered by moving the object slightly off-axis.

190. "Casimir Repulsion between Metallic Objects in Vacuum," M. Levin, A. P. McCauley, A. W. Rodriguez, M. T. Homer Reid, and S. G. Johnson, Phys. Rev. Lett. **105**, 090403 (4 pages) (2010). (A)

191. "A Diagrammatic Expansion of the Casimir Energy in Multiple Reflections: Theory and Applications," M. F. Maghrebi, Phys. Rev. D **83**, 045004 (12 pages) (2011). (I)

The latter reference verifies the repulsion by analytical calculation. Presumably, the same repulsion occurs for the Casimir-Polder force for an anisotropic molecule above a punctured conducting plate; calculations are currently underway to investigate this effect.

XIV. SELF-ENERGIES

We have already referred to several instances of Casimir self-energies, for example, for spheres and cylinders, Refs. [177], [178] [179], [180]. See also

192. "Casimir Energies for Massive Scalar Fields in a Spherical Geometry," M. Bordag, E. Elizalde, K. Kirsten, and S. Leseduarte, Phys. Rev. **D56**, 4896–4904 (1997). (A)

193. Casimir Energy for a Massive Fermionic Quantum Field with a Spherical Boundary, E. Elizalde, M. Bordag, and K. Kirsten, J. Phys. A **31**, 1743–1759 (1998). (A)

There was also extremely interesting results on the dimensional dependence of the Casimir effect, for example,

194. "Scalar Casimir Effect for a D-Dimensional Sphere," C. M. Bender and K. A. Milton, Phys. Rev. D **50**, 6547–6555 (1994). (A)

195. "Vector Casimir Effect for a D-Dimensional Sphere," K. A. Milton, Phys. Rev. D **55**, 4940–4946 (1997). (A)

Such calculations have somewhat obscure physical meaning, for although they are mathematically well-defined, it is difficult to see how they lead to an observable effect. For example, if a conducting sphere is cut in half and pulled apart, it will experience an attraction (due to the closest parts of the surface) not a repulsion. Nevertheless, the study of self-energies has remained an important part of Casimir studies. For a status report on the subject see Ref. [81]. Very recent work that shows intriguing systematics of the sign and magnitude of the Casimir self-stress is given in the following.

196. "Casimir Energies of Cylinders: Universal Function," E. K. Abalo, K. A. Milton, and L. Kaplan, Phys. Rev. D **82**, 125007 (12 pages) (2010). (I)

XV.   NEW EXPERIMENTAL DEVELOPMENTS

Within the last year, the first experiment was published measuring the Casimir-Polder force between an atom and a solid surface in the intermediate region between the nonretarded (vdW) and the retarded (CP) regime.

197. "Direct Measurement of Intermediate-Range Casimir-Polder Potentials," H. Bender, Ph. W. Courteille, C. Marzok, C. Zimmermann, and S. Slama, Phys. Rev. Lett. **104**, 083201 (4 pages) (2010). (I)

The following paper shows that the data agree best with a full QED calculation, and not with the retarded CP potential.

198. "Macroscopic QED—Concepts and Applications," S. Scheel and S. Y. Buhmann, Acta Phys. Slov. **58**, 675–809 (2008). (I)

There are new experiments that will test various aspects of Casimir-Polder forces, for example, between an atom and a complex surface.

199. "Nanowire Atomchip Traps for Sub-Micron Atom-Surface Distances," R. Salem, Y. Japha, J. Chabé, B. Hadad, M. Keil, K. A. Milton, and R. Folman, New J. Phys. **12**, 023039 (28 pp) (2010). (I)

They discuss a proposed measurement of atoms trapped in a Bose-Einstein condensate above a trapping wire on bilayer dielectric surface. And the CP interaction of a atom with a wall could be tested with atomic clocks:

200. "Mapping Out Atom-Wall Interaction with Atomic Clocks," A. Derevianko, B. Obreshkov, and V. A. Dzuba, Phys. Rev. Lett. **103**, 133201 (4 pages) (2009). (I)

Further work will allow us to calculate energy shifts of an atom near a layered microstructure:

201. "Casimir-Polder Interaction between an Atom and a Dielectric Slab," A. M. Contreras Reyes and C. Eberlein, Phys. Rev. A **80**, 032901 (10 pages) (2009). (I)

XVI. CONCLUSIONS

This Resource Letter is designed to guide the reader, whether a student or a researcher in another field who desires to become initiated in the subject, to the extensive literature on Casimir-Polder, van der Waals, and in general quantum vacuum forces between neutral atoms, molecules, and microscopic and macroscopic bodies. (For example, Physical Review lists nearly 800 citations of the 1948 Casimir-Polder paper, Ref. [41].) As one can tell, the field is undergoing a rapid period of development, with important innovations in both theory and experiment. It is nearly certain that within the next decade Casimir forces will give rise to practical nanotechnological applications.

But there is more. The current cosmological observations strongly suggest that 70% of the energy in the universe consists of "dark energy," which results in the observed cosmic acceleration. Although the errors are large, the equation of state of the dark energy, $w = -p/\rho$, in terms of the pressure and density of the dark energy, is consistent with the value

predicted if the dark energy were Einstein's cosmological constant, $w = -1$. If so, it is highly likely that the dark energy is due to quantum fluctuations of fields in the universe or extra dimensions.

For a Resource Letter on Dark Energy see

202. "Resource Letter DEAU-1: Dark Energy and the Accelerating Universe," E. V. Linder, Am. J. Phys. **76**, 197–204 (2008). (E)

For ideas of how dark energy might arise from quantum fluctuations, see for example,

203. "Constraints on Extra Dimensions from Cosmological and Terrestrial Measurements," K. A. Milton, Grav. Cosmol. **8**, 65–72 (2002). (I)

However, the particular model proposed there seems incompatible with the latest laboratory tests on gravity at short distance. For a survey of the long-standing difficulties in understanding the cosmological constant, see

204. "The Cosmological Constant Problem," S. Weinberg, Rev. Mod. Phys. **61**, 1–23 (1989). (I)

This is updated in

205. "The Cosmological Constant Problems," S. Weinberg, talk given at the 4th International Symposium on Sources and Detection of Dark Matter in the Universe (DM 2000) 23–25 February 2000, Marina del Rey, California, Proceedings, ed. D. B. Cline. (Berlin, Springer-Verlag, 2001); arXiv:astro-ph/0002387. (I)

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